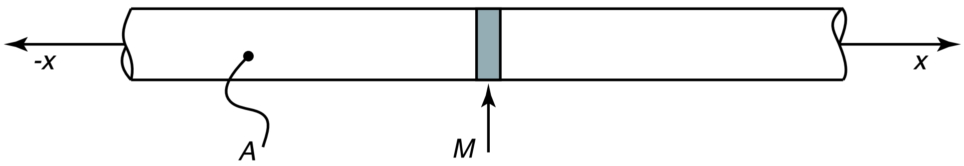
Exercise set 3

Problem 1



Consider the pipe section depicted in the figure above. A student injects 1.15g of Rhodamine-WT instantaneously and uniformly over the pipe cross-section ($A = 0.8\ cm^2$) at the point $x = 0$ and the time $t = 0$ (instantaneous point source). The pipe is filled with stagnant water.

We recall here the solution of the diffusion equation in one dimension (if needed, you may verify that it is a solution to the diffusion equation)

$$\rho(x,t) = \frac{\text{Cste}}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} \tag{1}$$

where we assume that the molecular diffusion coefficient is $D = 0.13 \cdot 10^{-4} cm^2/s$.

- What is the concentration at $x = 0$ at the time $t = 0$?
- What is the standard deviation of the concentration distribution $1s$ after injection?
- Plot the maximum concentration in the pipe, $C_{\max}(t)$, as a function of time over the interval $t = [0, 24h]$.
- How long does it take until the concentration over the region $x = \pm 1m$ can be treated as uniform? Define a uniform concentration distribution as one where the minimum concentration within a region is no less than 95% of the maximum concentration within that same region.

Problem 2

Consider the solution of the diffusion equation in one dimension $\frac{\text{Cste}}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}$. As in Problem 1 we have $\text{Cste} = M/A$ with M the mass injected at $x = 0$ and $t = 0$ and A the cross section. Verify that:

- The total mass M is conserved at each time t, i.e., $\int_{-\infty}^{\infty} \frac{\text{Cste}}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} dx = \text{Cste}\ \forall t$.
- The variance is given by $\sigma^2 = 2Dt$, i.e., $\frac{1}{\text{Cste}} \int_{-\infty}^{\infty} x^2 \frac{\text{Cste}}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}} dx = 2Dt\ \forall t$. Note that the factor $\frac{1}{\text{Cste}}$ ensures the proper normalization according to the previous point.

Hint: Using a change of variables the problem can be reduced to computing the integrals $\int_{-\infty}^{\infty} e^{-\alpha^2} d\alpha$ and $\int_{-\infty}^{\infty} \alpha^2 e^{-\alpha^2} d\alpha$.

Problem 3

In this problem we study numerically the symmetric one dimensional random walk with step length one using Matlab.

- Simulate various realizations of the symmetric one dimensional random walk. Use the function **random_walk_1d_plot**(STEP_NUM) with STEP_NUM the number of steps to take.
- Compute the average distance squared d^2_{average} as a function of the step number n. Use the function **random_walk_1d_simulation**(STEP_NUM,WALK_NUM) with STEP_NUM the number of steps to take in one test and WALK_NUM the number of times to repeat the walk to compute the average. The average distance squared is given by the blue curve. Compare this result with the prediction of the theory.
- Find the dependence on n of the maximal distance squared d^2_{max} at a step n for a large number of realizations (the maximum is taken over all the realizations). Use again the function **random_walk_1d_simulation**(STEP_NUM,WALK_NUM), the maximal distance squared is given by the green curve. Explain why the result is trivial.
- Consider several realizations of the random walk. Plot the histograms of the walkers spatial distribution at the step n and repeat the experiment for different total number of walkers. Use the function **random_walk_1d_histogram**(WALK_NUM). Iteration of the function allows to increase n (with fixed total number of walkers). Can you recognize a famous distribution when the number of walkers becomes large ?

Hint: You need to set the Path to the directory where the three .m files are located in order to use them.